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On Subgroups of ·O
II. Local Structure

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It is the purpose of this paper to produce a list of subgroups of $\cdot O$ with the following two properties: (a) The normalizer of any elementary abelian subgroup of $\cdot O$ is contained in a conjugate of one of the list. (b) No member of the list is conjugate to a subgroup of another member of the list.

Notation. As stated in [1] much of the notation is taken from [4–6]; for further familiarity with the group M_{24} the reader is referred to [2, 3, 7].

Now let the conjugacy classes in a group G be X, Y, Z.... We shall refer to an elementary abelian p-group as being *T-pure* if all its nontrivial elements belong to the class T.

We restrict out attention to a subset of the elementary abelian subgroups:

DEFINITION. An elementary abelian subgroup H of G will be said to be *relevant* if its normalizer is not contained in the normalizer of any nontrivial subgroup of it. (Thus every orbit of elements of H under the action of $N(H)$ must generate H.)

Note. $\cdot 1$ is the simple group obtained from $\cdot O$ by factoring out the central involution -1. We shall work in $\cdot O$ and define a relevant subgroup of $\cdot O$ to be the inverse image of a relevant subgroup of $\cdot 1$.

THE 2-LOCAL STRUCTURE OF ·O

$\cdot O$ contains three classes of involutions not in its center and one class of elements of order four whose squares are in the center. These are:

Element	Trace	Centralizer order in $\cdot O$	Shape in $\cdot 1$	Dimension of fixed space
B	-8	$2^{22} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^8 \cdot W(E_8)'$	8
C	0	$2^{18} \cdot 3^3 \cdot 5 \cdot 11$	$2^{11} \cdot M_{12}$	12
D	8	$2^{22} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^8 \cdot W(E_8)'$	16
F	0	$2^{15} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	$2^2 \times G_2(4)$	0

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Since the subgroup $N \cong 2^{12} \cdot M_{24}$ has odd index in $\cdot O$ any 2-subgroup is conjugate to a subgroup of N; we list the elements of N lying in the above classes:

I	ϵ_O	where $O \in \mathscr{C}_8$		D
II	ϵ_C	where $C \in \mathscr{C}_{12}$		C
III	$\epsilon_{O+\Omega}$	where $O \in \mathscr{C}_8$		B
IV	$\pi\epsilon_C$	where $\pi \sim 1^8 \cdot 2^8$,	$C \cap F(\pi) = \varnothing$, $C^\pi = C$	D
V	$\pi\epsilon_C$	where $\pi \sim 1^8 \cdot 2^8$,	$C \cap F(\pi) = F(\pi)$, $C^\pi = C$	B
VI	$\pi\epsilon_C$	where $\pi \sim 1^8 \cdot 2^8$,	$\lvert C \cap F(\pi)\rvert = 4$, $C^\pi = C$	C
VII	$\pi\epsilon_C$	where $\pi \sim 2^{12}$,	$C^\pi = C$	C
VIII	$\pi\epsilon_C$	where $\pi \sim 2^{12}$,	$C^\pi = C + \Omega$	F

Note. The symbol $F(\pi)$ denotes the octad fixed by the $1^8 \cdot 2^8$ involution π of M_{24} and ϵ_D denotes a reflection along the coordinate vectors $\mathbf{v}_i$ for $i \in D$. Any element of N can be written $\pi\epsilon_D$, where π is a permutation of M_{24} and D is a $\mathscr{C}$-set—and this is an involution iff $\pi^2 = 1$ and $D^\pi = D$.

THE LEECH LATTICE Λ, THE FACTOR SPACE $\Lambda/2\Lambda$ AND THE CROSSES

If we take the Leech lattice and factor out the sublattice 2Λ we get a vector space $\Lambda/2\Lambda$ of dimension 24 over the field GF_2. It turns out that the $2^{24} - 1$ elements of this space may be taken to be the 2-vectors, the 3-vectors, and the sets of 24 mutually orthogonal 4-vectors (and their negatives) which we are referring to as crosses. This fact is Theorem 10 of the Conway paper [5] and its proof is simply by counting. We shall call the 24 pairs, a vector and its negative, *diameters* of the cross. Of course every 4-vector belongs to a unique cross since the relation "is congruent modulo 2Λ" is an equivalence relation on the set of vectors. Now the cross containing the 4-vector $8\mathbf{v}_\infty$ consists of the 48 vectors of this shape; it is known as the *standard cross* and its stabilizer is the group N. There are, moreover, two crosses associated with each sextet D; these are denoted by D^e and D^o where the 4-vectors of D^e are of shape $4^4 \cdot 0^{20}$ with nonzero entries on a tetrad of D and an *even* number of positive coefficients. D^o is similarly defined but the 4-vectors now have an *odd* number of positive coefficients.

We recall that the seven refinements of the *MOG* trio to sextets are:

o o	o o	o o	o x	o x	o x	o x
o o	x x	x x	o x	o x	x o	x o
x x	o o	x x	o x	x o	o x	x o
x x	x x	o o	o x	x o	x o	o x
A	B	C	D	E	F	G

(the tetrads being repeated in each brick).

Thus D^{e} would denote the cross containing the 4-vector

$$\begin{array}{|cc|cc|cc|}\hline 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}_{\,M}$$

We observe that the normal subgroup 2^{12} in N (usually denoted by E) consists of all those elements of N which fix every diameter of the standard cross, X. We ask how many crosses the different involutions of $\cdot O$ fix *diametrically*.

We find:

(i) A B- or D-involution fixes 135 crosses diametrically, explaining why $|\, C_{\cdot O}(D): C_N(D)| = 135$. [Choosing ϵ_O for $O \in \mathscr{C}_8$].

(ii) A C-involution fixes *precisely one cross diametrically*, and so $C_{\cdot O}(C) = C_N(C)$ ($\cong 2^{12} \cdot M_{12}$). (Choosing ϵ_C for $C \in \mathscr{C}_{12}$.)

(iii) An F-involution fixes no cross diametrically.

Fact (ii) will enable us to deal with all elementary abelian 2-groups which contain C-involutions. For each type of C-involution occurring in N we list its *associated cross* and for the C-involution x we denote it by $\chi(x)$. Thus:

$\chi(\epsilon_O)$ for $C \in \mathscr{C}_{12}$ is the standard cross X.

$\chi(\pi\epsilon_C)$ for $\pi \sim 2^{12}$ is $S(\pi)^{\mathrm{o}}$ or $S(\pi)^{\mathrm{e}}$ where $S(\pi)$ is the unique sextet fixed tetrad-wise by π.

$\chi(\pi\epsilon_C)$ for $\pi \sim 1^8 \cdot\cdot 2^8$ is D^{o} or D^{e} where D is the sextet defined by the tetrad $C \cap F(\pi)$.

In particular we notice that all the crosses are of the sextet-type or X.

The Bilinear Form on $\Lambda/2\Lambda$

If we take the ordinary inner product on Λ, normalize by dividing by 16, and read modulo 2 we obtain the natural bilinear form on $\Lambda/2\Lambda$. The crosses orthogonal to the standard cross, X, are clearly just those of sextet-type since every 4-vector must have all coefficients divisible by 4. Further: two sextet-type crosses are orthogonal if, and only if, every tetrad of one sextet intersects every tetrad of the other in an even number of points (see [3, Lemma 3] for intersection of sextets). We call the two-dimensional subspace of $\mathscr{C}^*$ generated by two such sextets an *even line of sextets*. Now the subspace of $\Lambda/2\Lambda$ spanned by a set of mutually orthogonal crosses must be isotropic and so all crosses in it are orthogonal to one another.

The Relevant 2-Subgroups of $\cdot O$ which Contain C-Involutions

Let H be such a subgroup and assume $\epsilon_M \in H$ where $M \in \mathscr{C}_{12}$. Thus $H \leqslant E \cdot M_{12} \cdot 2 \leqslant N$. Let $\chi(H)$ denote the set of crosses associated with C-involutions of H. $X \in \chi(H)$ and all other members of $\chi(H)$ are of the sextet-type. Plainly $N._O(H)$ must permute the set $\chi(H)$; we consider the orbit of $N._O(H)$ which contains X, $X^{N(H)}$. Since X is orthogonal to every member of $\chi(H)$ every member of $X^{N(H)}$ must be orthogonal to every other member. But $N._O(H) \leqslant \mathrm{Stab}._O\{X^{N(H)}\} \leqslant \mathrm{Stab}._O\langle X^{N(H)}\rangle$ and by the above any two crosses of $\langle X^{N(H)}\rangle$ are orthogonal. It is now clear that we are looking for sextet subspaces of $\mathscr{C}^*$ all of whose two-dimensional subspaces are even lines. It is an easy matter to find these using the *MOG*; they are:

(a) A single sextet, D, say.

(b) An even line e.g. the sextets A, D and E.

(c) Ref_3—consisting of the seven refinements of a trio, say A, B, C, D, E, F, G.

(d) Inv_3—consisting of the seven sextets which are given by the unions of two transpositions of a $1^8 \cdot 2^8$ involution of M_{24}.

Thus there are five possibilities for the subspace $\langle X^{N(H)}\rangle$:

(0) X;

(a) X, D^e, D^o;

(b) $X, A^e, A^o, D^e, D^o, E^e, E^o$;

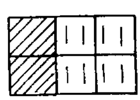
(c) X, L^e, L^o, for $L = A, B, C, D, E, F, G$;

(d) X, L^e, L^o for $L = A, D, E, H, I, J, K$.

Where A, B, C, D, E, F, G represent the usual refinements of the *MOG* trio while H, I, J, K represent (here only) the sextets:

$$\begin{array}{|cc|cc|cc|}\hline \mathrm{x} & \mathrm{x} & 1 & 2 & 1 & 2\\ \mathrm{o} & \mathrm{o} & 1 & 2 & 1 & 2\\ \hline \mathrm{o} & \mathrm{x} & 3 & 4 & 3 & 4\\ \mathrm{x} & \mathrm{o} & 3 & 4 & 3 & 4\\ \hline\end{array}_{M} \quad \begin{array}{|cc|cc|cc|}\hline \mathrm{x} & \mathrm{x} & 1 & 2 & 2 & 1\\ \mathrm{o} & \mathrm{o} & 1 & 2 & 2 & 1\\ \hline \mathrm{x} & \mathrm{o} & 3 & 4 & 4 & 3\\ \mathrm{o} & \mathrm{x} & 3 & 4 & 4 & 3\\ \hline\end{array}_{M} \quad \begin{array}{|cc|cc|cc|}\hline \mathrm{x} & \mathrm{o} & 1 & 2 & 3 & 4\\ \mathrm{o} & \mathrm{x} & 1 & 2 & 3 & 4\\ \hline \mathrm{o} & \mathrm{o} & 3 & 4 & 1 & 2\\ \mathrm{x} & \mathrm{x} & 3 & 4 & 1 & 2\\ \hline\end{array}_{M} \quad \begin{array}{|cc|cc|cc|}\hline \mathrm{x} & \mathrm{o} & 1 & 2 & 3 & 4\\ \mathrm{o} & \mathrm{x} & 1 & 2 & 3 & 4\\ \hline \mathrm{x} & \mathrm{x} & 4 & 3 & 2 & 1\\ \mathrm{o} & \mathrm{o} & 4 & 3 & 2 & 1\\ \hline\end{array}_{M}$$

respectively. We are considering the Inv_3 space obtained from the involution

$\sigma =$  M

Now the element ζ_D of [4], where D is any tetrad of the sextet D preserves each of the last four sets of crosses. In fact its action is:

$\zeta_D : (XD^o)(L^e)(A^oE^o)(B^oF^o)(C^oG^o)(H^oJ^o)(I^oK^o)$ where L runs through $A - K$.

Adjoining the element ζ_D to the subgroups of N preserving the above sets of crosses thus gives their full stabilizing groups. We see that these are:

(0) N

(a) $(\underbrace{2^{11}}_{\text{in } E} \cdot \underbrace{\text{sextet group}}_{\text{in } M_{24}}) \cdot \underbrace{S_3}_{\text{action on the crosses}}$

(b) $(\underbrace{2^{10}}_{\text{in } E} \cdot \underbrace{2^6 \cdot SxV_4}_{\text{in trio gp. of } M_{24}}) \cdot \underbrace{L_3(2)}_{\text{action on the crosses}}$

(c) $(\underbrace{2^9}_{\text{in } E} \cdot \underbrace{2^6 \cdot S_3}_{\text{in trio gp. of } M_{24}}) \cdot \underbrace{L_4(2)}_{\text{action on the crosses}}$

(d) $(\underbrace{2^9}_{\text{in } E} \cdot \underbrace{2^4 \cdot 2^3}_{\text{in } C_{M_{24}}(\pi)}) \cdot \underbrace{L_4(2)}_{\text{action on the crosses}}$

(where $\pi \sim 1^8 \cdot 2^8$).

The group in (b) is contained in the group in (c) since anything fixing the seven crosses of (b) fixes the 15 of (c).

The group in (d) centralizes the involution $\epsilon_{F(\sigma)}$ and so is not maximal. 2^{22} divides the order of the groups in both (a) and (c) and since they both contain the whole group E they cannot lie in any copy of N. Nor is there an involution visibly centralized in either case, and so we conclude that we do not know proper subgroups of $\cdot O$ containing them.

We have thus thus shown:

LEMMA 2.2. *The normalizer of any elementary abelian subgroup of $\cdot O$ which contains C-involutions is contained in a conjugate of* (0), (a), *or* (c) *above or centralises an involution.*

We now prove two simple lemmas which clarify the problem.

LEMMA 2.3. *An XY-relevant 2-group in $\cdot O$ contains fourgroups of types XXY and YYX.*

Proof. If the product of any two X-involutions is an X-involution then the set of all X-involutions in the group forms an invariant subgroup, thus the original subgroup could not have been relevant.

COROLLARY. *No relevant subgroup of $\cdot O$ contains BD and F-involutions but no C-involutions*

Proof. Any such would have to contain fourgroups of shape *BBF* or *BDF*. But if x is an F-involution, y a B-involution, then $(xy)^2 = 1$ implies $y^x = -y$. Contradiction since $-y$ is a D-involution. Thus:

LEMMA 2.4. *Any relevant 2-subgroup not containing C-involutions is BD-pure or F-pure.*

THE BD-PURE SUBGROUPS

The BD-subgroups of E correspond to the subspaces of $\mathscr{C}$ which contain no dodecads. We exhibit these in fig. 2.1, where the $\mathscr{C}$-sets in each case are just those unions of blocks which *are* $\mathscr{C}$-sets (we may assume $\epsilon_\Omega = -1$ belongs to our subgroup since its adjunction can only increase the normalizer). Note that an n-dimensional subspace contains $2^{n-1} - 1$ octads.

Now any BD-pure subgroup of order 8 not in E may be assumed to contain (1) and, under conjugation by elements of E, a $1^8 \cdot 2^8$ permutation of M_{24}, π. Since

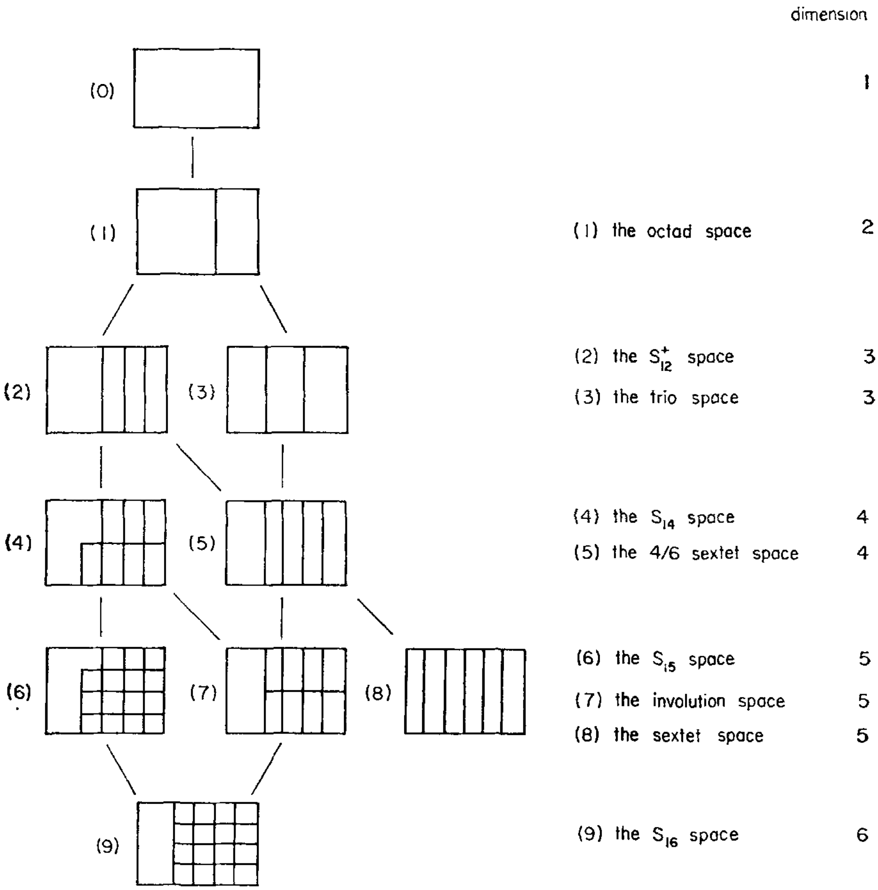
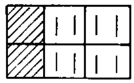
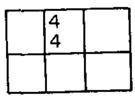


FIG. 2.1. The subspaces of $\mathscr{C}$ which contain no dodecads.

the octad in (1) must be fixed by π and must equal or be disjoint from $F(\pi)$, we see that in every case there is an even sextet-type cross D^e fixed diametrically by the subgroup, i.e., *any BD-subgroup of order* 8 *is conjugate to a subgroup of E.* But similarly, by adjoining a $1^8 \cdot 2^8$ permutation to (2) and (3) in turn, we see that *any BD-subgroup of order* 16 *fixes a cross diametrically and is thus conjugate to a subgroup of E.* The same argument applies to (5) but there is a unique involution

$\sigma =$ 

which may be added to (4) to give a new BD-subgroup of order 32. This group's normalizer is contained in $\cdot 2$, however, as it fixes the vector:



Nor may this group be extended except in E as σ was unique. (6) cannot be extended except to (9); (8) and (9) cannot be extended at all without giving C-involutions. (7) *can* be extended by the addition of σ.

DEFINITION. We shall say two D-involutions are *joined* if their product is another D-involution.

We look at the graphs defined on the D-involutions in each of the above cases. (5), (7), (9) and $\langle (7), \sigma \rangle$ have unique isolated points and so cannot be relevant. This leaves (1), (2), (3), (4), (6), and (8) whose normalizers must be considered; to this end ask which crosses they fix diametrically.

(2) and (8) fix only X, D^e and D^o (see above).

(3) fixes X, L^e and L^o for $L = A - G$.

(4) and (6) fix only X.

We have thus proved:

LEMMA 2.5. *The normalizer in $\cdot O$ of any BD-pure subgroup is contained in one of*: (0), (a), *or* (c) *of Lemma* 2.2, *the stabilizer of a* 2*-vector, or the centralizer of a D-involution.*

THE F-PURE SUBGROUPS

LEMMA 2.6. *A maximal F-pure subgroup of $\cdot O$ is a "fourgroup," i.e., a quaternion group all of whose* 4*-elements are "F-involutions."*

Proof. Suppose x, y, z are F-involutions generating an F-pure subgroup of order 16 in $\cdot O$. Then $(xy)^2 = -1 \Rightarrow x \cdot y^{-1} \cdot x \cdot y = 1 \Rightarrow x^y = -x$. So $x^{yz} = x$ and xyz cannot be an F-element. Contradiction.

We may now compute the character sum $l_i^{ii} = (h_i^2/n) \sum(\chi_i{}^t)^3/\chi^j(1)$ to find the number of ways a particular F-involution can be expressed as the product of two F-involutions; we see that the sum of the reciprocals of the centralizers of F-pure fourgroups is $389/2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$. We claim that there are three classes of F-pure fourgroups with centralizer orders $2^{13} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$, $2^{13} \cdot 3 \cdot 5$, and $2^7 \cdot 3^3 \cdot 7$. To prove this it will suffice to produce three plainly nonconjugate fourgroups with centralizer orders *at most* the above.

Now an F-element fixes $2^{12} - 1$ crosses, i.e., in its action on $\Lambda/2\Lambda$ it has a fixed space of dimension 12. If we denote the F-pure fourgroups by F_i $(i = 1, 2, 3)$ we see that $| C._O(F_i): C_N(F_i)|$ = the number of crosses in the orbit of $C._O(F_i)$ which contains X. As usual the numbers $| C_N(F_i)|$ are readily found.

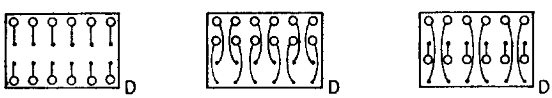
We now give an example of each type of fourgroup—the subscript D will mean that the diagram refers to the DOG array Fig. 2.2. In all cases the element consists of the permutation followed by negation on the ringed $\mathscr{C}$-set.

∞	14	17	11	19	22
0	18	4	2	6	1
3	8	16	13	9	12
15	20	10	7	5	21

D

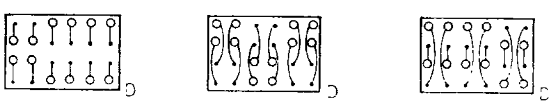
FIG. 2.2. The DOG array which admits a group $L_2(5)$ on its columns, fixing its rows, and a group A_4 on its rows fixing its columns.

F_1:



Here we find that any cross fixed by one of the three elements is fixed by all three. Further the centralizer in N of any one of them preserves the group F_1. Thus the *maximum* order for $C._O(F_1)$ is $2^{13} \cdot 3 \cdot 5 \times 63.65$.

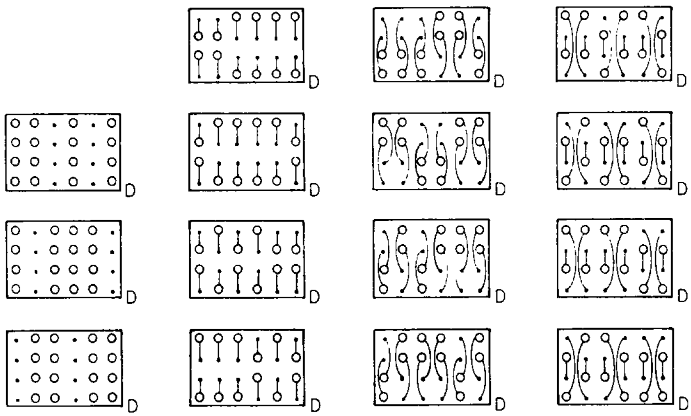
F_2:



This differs from our F_1 only in that the reflections have been altered by a $\mathscr{C}$-set which is a union of tetrads of the sextet defined by each of the permutation parts. There are only $\binom{6}{2}$ possible choices for this $\mathscr{C}$-set (modulo -1) and so $| C_N(F_1): C_N(F_2)| = 15$, i.e., $| C_N(F_2)| = 2^{13}$.

We now adjoin to F_2 the unique F_1's associated with its three elements to obtain a group which must be normalized by the centralizer of the original F_2.

It turns out that this group is in fact of order 16 and contains a unique (and thus invariant) BBB-subgroup. We record the 16 elements here:



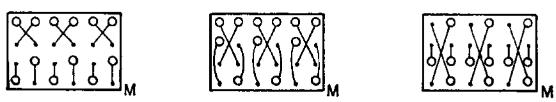
(The columns of F-elements are cosets of the BBB-group and the rows are all F_1's.)

Now we see that the BBB-group fixes just 15 crosses diametrically and in this instance one of the 15 is X. Thus there are at most 14 cosets of the group we seek outside N, i.e., the *maximal* order of $C._O(F_2)$ is $2^{13} \cdot 15$.

We note in passing that this group is no longer of interest since it is contained in $C._O(BBB)$.

The complete space of crosses fixed by an F_2 is, in fact, of dimension 8 and the 255 crosses split into two orbits of lengths 15 and 240 under the action of the centralizer; so F_2-groups occur in N in two different ways.

F_3 :



This group fixes just 63 crosses and so has fixed space of dimension 6. Since $C_N(F_3) \simeq 2^4 \cdot 2^3 \cdot 3$, we see that $| C._O(F_3)| \leqslant 2^7 \cdot 3 \cdot 63$. We have now shown:

(a) there are just three classes of F-fourgroup,

(b) their centralizers have the orders asserted, and

(c) their centralizers are transitive on the sets of crosses mentioned.

The normalizer of the group F_1 is in fact a group $2(A_4 \times G_2(4)) \cdot 2$ where the $2A_4$ is the quaternion group F_1 with a f.p.f. 3-element (class A) normalizing it.

If we adjoin to an F_3 group the three F_1's associated with it we find they generate a group $2A_6$ and this is centralized (modulo the center) by a $U_3(3)$.

Thus an interesting and possibly maximal subgroup of $\cdot O$ containing the normalizer of an F_3 is $2(A_6 \times U_3(3)) \cdot 2$.

THEOREM 2.1. *We have now shown that the normalizer of any elementary abelian 2-group of $\cdot O$ is contained in a conjugate of one of:*

(i)	$C(B)$	$2^8 \cdot W(E_8)'$
(ii)	$\mathrm{Stab}(X)$	$2^{11} \cdot M_{24}$
(iii)	Stab(2-*vector*)	$\cdot 2$
(iv)	N(*sextet space of* $\mathscr{C}$)	(2^{10} · *sextet group in* M_{24}) $\cdot S_3$
(v)	N(*trio space of* $\mathscr{C}$)	$2^{8+6} \cdot (S_3 \times L_4(2))$
(vi)	$N(F_1)$	$(A_4 \times G_2(4)) \cdot 2$ (*see Appendix*)
(vii)		$(A_6 \times U_3(3)) \cdot 2$ (*see Appendix*)

Note. All shapes are given in $\cdot 1$, i.e., modulo the center.

3-LOCAL STRUCTURE OF $\cdot O$

There are four classes of elements of order 3 in $\cdot O$:

Element	Trace	Centralizer order in $\cdot O$	Shape in $\cdot 1$	Dimension of fixed space	Fixed space mod 2Λ
A	-12	$2^{14} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$3 \cdot Sz$	0	—
B	-3	$2^8 \cdot 3^9 \cdot 5$	$2^{1+4} \cdot 2U_4(2)$	6	$2^{27} \cdot 3^{36}$
C	0	$2^7 \cdot 3^5 \cdot 5 \cdot 7$	$3 \times A_9$	8	$2^{120} \cdot X^{135}$
D	6	$2^9 \cdot 3^8 \cdot 5 \cdot 7$	$3^2 \cdot U_4(3) \cdot 2$	12	$2^{378} \cdot 3^{2016} \cdot X^{1701}$

LEMMA 3.1. *The only relevant 3-group containing C-elements has order 3.*

Proof. In $C(C) \cong 3 \times A_9$ there are no C-elements in the A_9, but all diagonal 3-elements are of C-type (see Appendix). Thus in any elementary abelian 3-group containing C-elements the (well-defined) complement of one such is characteristic.

LEMMA 3.2. *Any relevant 3-group containing D-elements may be assumed to be generated by A-elements.*

Proof. The 12-dimensional subspace fixed by a D-element, d, contains 378 2-vectors which fall into 126 blocks of 3—two vectors being in the same block if they are orthogonal to the same set of vectors. The orthogonal complement to this 12-space is of the same kind and defines a further D-element, d^*, fixing it. The full action on these two 12-spaces is $3^2 \cdot (U_4(3) \cdot D_8)$—where both actions

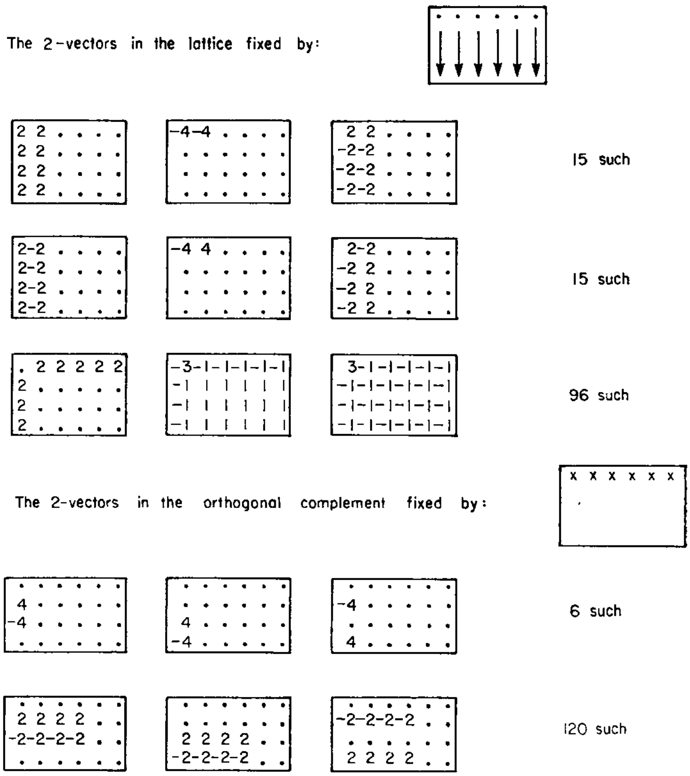


FIGURE 3.1

on 126 letters are realized and an outer AM interchanges the two sets (see Fig. 3.1).

Now let H be an elementary abelian 3-group containing a D-element d, and let the *mate* of d be d^*. Plainly since every element of H commutes with d, it must commute with d^* (it cannot invert d^* being of order 3), i.e., $\langle H, d^*\rangle$ is elementary abelian. In this way we can extend H to an elementary abelian group H^* which contains the mates of all the D-elements of H. But clearly $N(H^*) \geqslant N(H)$—since H defines H^* uniquely. But H^* contains A-elements and the subgroup generated by these is characteristic.

N.b. Figure 3.1 shows the 378 2-vectors fixed by a $1^6 \cdot 3^6$ permutation of M_{24} together with the complementary 378 2-vectors. We have denoted the D-element corresponding to the complementary space by x's in the coordinate positions fixed by the permutation, i.e., a nonspecial hexad S. The action of this element, which is closely related to ζ_T of [4], is

$$\mathbf{v}_i \to \mathbf{v}_i - (1/2)\mathbf{v}_T \qquad \text{if } i \notin S,$$
$$\mathbf{v}_i \to (1/2)\mathbf{v}_T - \mathbf{v}_i \qquad \text{if } i \in S, \text{ where } T \text{ is the tetrad containing } i$$

and congruent to S mod $\mathscr{C}$. Note that this element is completely defined by S to within inversion and, having order 3, trace 6, is D-type.

We may now restrict out attention to:

(i) elementary abelian 3-groups generated by A-type elements,

(ii) elementary abelian 3-groups containing only B-type elements.

(i) If we take a triangle of shape 333 in the Leech lattice and consider which other 333-triangles have the same center, we find that there are just 11 more. The group fixing these 12 triangles as a set is known as! 333 and has the shape $3^6 \cdot 2M_{12}$—where the 3^6 is an elementary abelian group fixing all the triangles but rotating them. We can observe that this group contains the full 3-part of $\cdot O$. Moreover when we view this group in the notation of the complex Leech lattice—where the 12 triangles become 12 coordinate vectors x_∞, $x_0, \ldots, x_{10}$ and positive rotation of the triangles becomes multiplication of the corresponding x_i by ω (the cube root of 1)—we see that there are just 12 (together with their inverses) in the 3^6 which multiply all the coordinate vectors by ω or $\bar{\omega}$ (i.e., which rotate all the triangles). We choose ω_∞ to rotate all triangles positively, and ω_k to rotate x_{N-k} positively and x_{Q-k} negatively. (Here Q denotes the set of squares mod 11 and N nonsquares mod 11 together with ∞. $x_S = \{x_s\,;\, s \in S\}$. Thus:

ω_∞	$\infty 0123456789X$
ω_0	$\infty 2678X - 013459$
ω_1	$\infty 15679 - 02348X$
	$\cdots$
ω_X	$\infty 03789 - 12456X$

showing which of the x_i are multiplied by ω and which by $\bar{\omega}$. Here X stands for 10. Note that the x_i's and the ω_i's correspond to the two permutation representations of M_{12} on 12 letters. For $2M_{12}$ in more detail see [6].

These elements have trace $12(\omega + \bar{\omega}) = -12$ and are thus A-type; they can be seen to be the only 3-elements in the whole group $3^6 \cdot 2M_{12}$ having this trace. (The trace of an element having nontrivial image in the M_{12} will be 0 when this image is of shape 3^4 and when it is of shape $1^3 \cdot 3^3$ it will be determined by its action on the three coordinate vectors it leaves fixed, so can only be -3, 0, or 6). Thus there are only 12 A-elements in a Syl_3 subgroup; these form a mutually commuting set, and any set of less than six mutually commuting A-elements is conjugate to any other.

We describe the 3^6 in more detail:

Name	Shape	Trace	Number
1	1^{12}	24	1
$\pm w_i$	ω^{12}	-12	24
$\pm w_i \pm w_j$	$w^6 \cdot 1^6$	6	264
$\pm w_i \pm w_j \pm w_k$	$\omega^6 \cdot 1^3 \cdot \bar{\omega}^3$	-3	440
(Four ways)			729

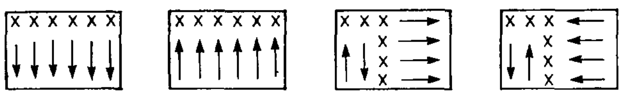
Table 3.1 shows the four ways in which an element of the fourth type may be written.

Now any six of the ω_i's which do not form a special hexad generate the whole of the 3^6; thus we are interested in the normalizers of the groups generated by one, two, three, four, or five commuting A-elements.

One gives the group $(3Sz) \cdot 2$

Two gives the group of order 3^2 defined by a D-element and its mate. The full group is $3^2 \cdot (U_4(3) \cdot D_8)$.

For three we abbreviate w_i to i, and $-w_i$ to $\bar{\imath}$ and ask in how many ways the 3^3-group $\langle \infty, 0, 1 \rangle$ extends to a 3^6. From the note following Lemma 3.2 we see that our elements ζ_S followed by a $1^6 \cdot 3^6$ permutation fixing S give A-elements. The following are four mutually commuting A-elements:



∞ $\bar{0}$ 1 $\bar{2}$

We name them as shown above.

Now ∞ 0 and ∞ 1 fix the three triangles x_∞, x_6, and x_7, i.e., they fix the nine vectors constituting these triangles. We find that there are just 36 3-vectors fixed by our ∞ 0 and ∞ 1 from above, and these fall into four nines (see Fig. 3.2) showing that there are four ways that a 3^3-group can be extended to a 3^6. (Note that the triangles are well defined by the action of the A-elements ∞, 0, and 1 in our 3^3.)

Thus $| N_{\cdot 1}(3^3) \colon N_{3^1 \cdot 2M_{12}}(3^3)| = 4$, and $N_{3^1 \cdot 2M_{12}}(3^3) = 3^6 \cdot 2M_9 \cdot S_3$.

So the normalizer of the 3^3-group generated by three commuting A-type 3-elements is soluble of order $2^7 \cdot 3^9$. We note in passing that the 3^3 contains just four B-elements (and their inverses) and the four 6-dimensional fixed spaces corresponding to these defines a decomposition of the 24-space which is preserved by the above normalizer.

Now a set of four mutually commuting A-elements extends uniquely to a set of 12 and so the normalizer of a group generated by 4, 5, or 6 commuting A-elements is contained in $3^6 \cdot 2M_{12}$.

(ii) The fixed space of a B-element is the $\mathscr{S}$-lattice $2^{27} \cdot 3^{36}$ of [1]. We give here the 27 2-vectors fixed by our element $\infty 012$ (see Fig. 3.3).

It remains to discover all possible classes of pure B-group. We may readily verify that the four $\mathscr{S}$-lattices corresponding to any B-pure 3^2-group must span the 24-dimensional space (by a consideration of the eigenspaces of each of the elements) and thus observe that a maximal B-pure group has order 3^2. Now in the 3^{1+4} of the centralizer of a B-element all elements except the central ones must be of D-type (they fix more than a six-dimensional subspace and 2-vectors). Our 3^2-

TABLE 3.1[a]

∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞
6789x012345	789x0123456	x0123456789	89x01234567	23456789x01
789x0123456	x0123456789	89x01234567	23456789x01	6789x012345
0123456789x	0123456789x	0123456789x	0123456789x	0123456789x
3456789x012	9x012345678	56789x01234	456789x0123	123456789x0
456789x0123	123456789x0	3456789x012	9x012345678	56789x01234
−123456789x0	−3456789x012	−9x012345678	−56789x01234	−456789x0123
−56789x01234	−456789x0123	−123456789x0	−3456789x012	−9x012345678
−9x012345678	−56789x01234	−456789x0123	−123456789x0	−3456789x012
−23456789x01	−6789x012345	−789x0123456	−x0123456789	−89x01234567
−89x01234567	−23456789x01	−6789x012345	−789x0123456	−x0123456789
−x0123456789	−89x01234567	−23456789x01	−6789x012345	−789x0123456
123456789x0	3456789x012	9x012345678	56789x01234	456789x0123
56789x01234	456789x0123	123456789x0	3456789x012	9x012345678
−789x0123456	−x0123456789	−89x01234567	−23456789x01	−6789x012345
6789x012345	789x0123456	x0123456789	89x01234567	23456789x01
x0123456789	89x01234567	23456789x01	6789x012345	789x0123456
−3456789x012	−9x012345678	−56789x01234	−456789x0123	−123456789x0
∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞
89x01234567	23456789x01	6789x012345	789x0123456	x0123456789
−456789x0123	−123456789x0	−3456789x012	−9x012345678	−56789x01234
0123456789x	0123456789x	0123456789x	0123456789x	0123456789x
9x012345678	56789x01234	456789x0123	123456789x0	3456789x012
−23456789x01	−6789x012345	−789x0123456	−x0123456789	−89x01234567
3456789x012	9x012345678	56789x01234	456789x0123	123456789x0
56789x01234	456789x0123	123456789x0	3456789x012	9x012345678
−89x01234567	−23456789x01	−6789x012345	−789x0123456	−x0123456789
∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞
x0123456789	89x01234567	23456789x01	6789x012345	789x0123456
−0123456789x	−0123456789x	−0123456789x	−0123456789x	−0123456789x
123456789x0	3456789x012	9x012345678	56789x01234	456789x0123
9x012345678	56789x01234	456789x0123	123456789x0	3456789x012
−789x0123456	−x0123456789	−89x01234567	−23456789x01	−6789x012345
23456789x01	6789x012345	789x0123456	x0123456789	89x01234567
6789x012345	789x0123456	x0123456789	89x01234567	23456789x01
−456789x0123	−123456789x0	−3456789x012	−9x012345678	−56789x01234
56789x01234	456789x0123	123456789x0	3456789x012	9x012345678
789x0123456	x0123456789	89x01234567	23456789x01	6789x012345
∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞	∞∞∞∞∞∞

[a] To be read vertically in blocks of four. Other terms obtained by negation.

TABLE 3.1 *(continued)*

3456789x012	9x012345678	56789x01234	456789x0123	123456789x0
6789x012345	789x0123456	x0123456789	89x01234567	23456789x01
−89x01234567	−23456789x01	−6789x012345	−789x0123456	−x0123456789
456789x0123	123456789x0	3456789x012	9x012345678	56789x01234
9x012345678	56789x01234	456789x0123	123456789x0	3456789x012
−0123456789x	−0123456789x	−0123456789x	−0123456789x	−0123456789x
123456789x0	3456789x012	9x012345678	56789x01234	456789x0123
x0123456789	89x01234567	23456789x01	6789x012345	789x0123456
−23456789x01	−6789x012345	−789x0123456	−x0123456789	−89x01234567

The four sets of 3 triangles fixed by ∞0 and ∞1.

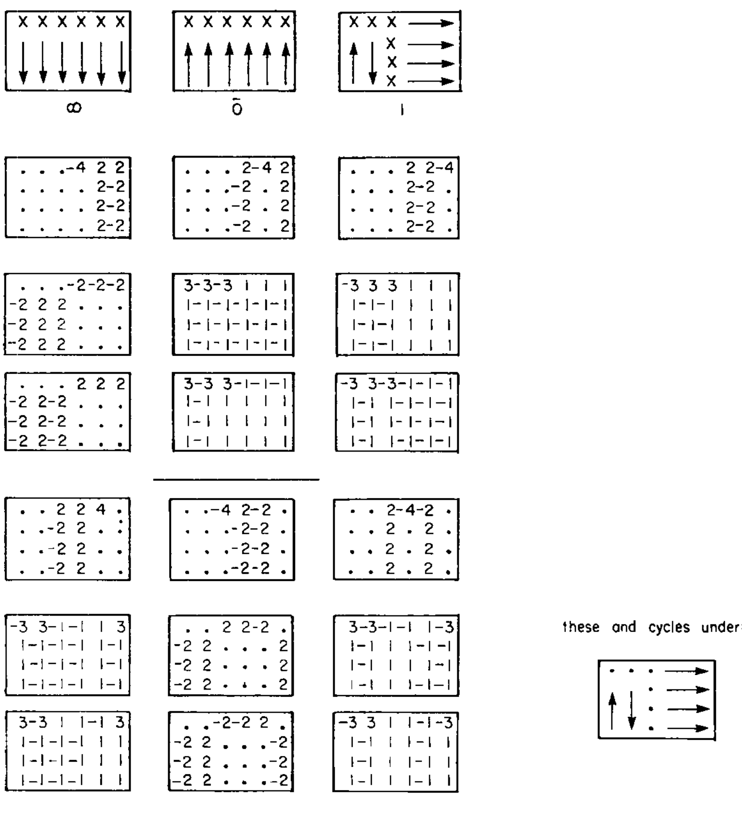


FIGURE 3.2

The 27 2-vectors in the S-lattice fixed by:

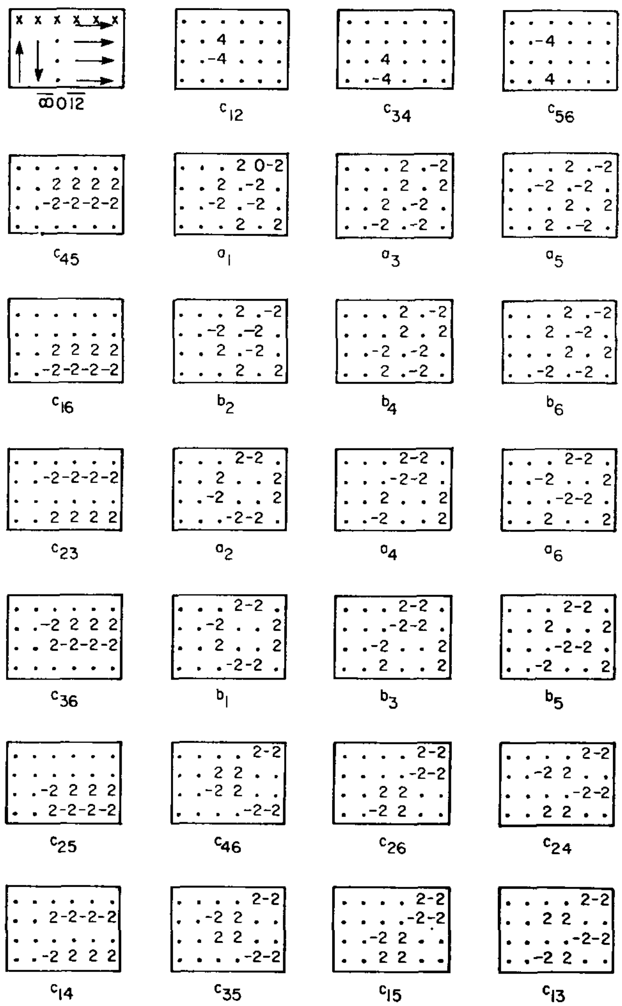


FIGURE 3.3

group must therefore contain elements with nontrivial image in the $U_4(2)$ and this image must fix no vector in the fixed space of the original element. From the character table of $U_4(2)$ we find that there are just two classes of 3-elements satisfying these requirements (they fuse in $U_4(2) \cdot 2$). To each such element there correspond 81 inverse images in $3^4 \cdot U_4(2)$ and it remains to discover the orbits of the stabilizer of such a coset on its members. But the classes of 3-elements of $U_4(2)$ which take the value -3 on the six-dimensional representation have length 40—and the members of such a class correspond 1–1 with the

40 one-dimensional subspaces in a four-dimensional space over GF_3, i.e., with the 40 subgroups of order 3 of the 3^4. We wish to know the action of the stabilizer in $2U_4(2)$ of one of the 80 *vectors* on the remainder. Since the norm of the corresponding permutation character is four we are able to deduce that the orbits are:

$$1 + 1 + 24 + 54.$$

The last inverse image corresponds to the unit element (times the representative in $U_4(2)$ that we started with). We must now consider the action of the 3^4 on these orbits—plainly there are elements in the 3^4 which are not fixed by the representative in $U_4(2)$ and so conjugation of this representative by one such must lead to some fusion. It is now clear that the three 1-orbits must fuse to give: $3 + 24 + 54$. Addition of an element in the 3-orbit does not lead to a B-pure subgroup (it is of type $ABBD$) but both of the others lead to B-pure 3^2-subgroups. The normalizers of these are soluble groups of orders $2^5 \cdot 3^9$ and $2^7 \cdot 3^8$ respectively in $\cdot 1$.

In conclusion we note that we have seen three distinct ways in which the 24-dimensional space may be decomposed as the direct sum of four 6-dimensional subspace each of whose intersection with the Leech lattice is an $\mathscr{S}$-lattice $2^{27} \cdot 3^{36}$.

THEOREM 3.1. *Any* 3-*local subgroup of* $\cdot O$ *is conjugate to a subgroup of one of the following*:

(i)	$N(A)$	$(3Sz) \cdot 2$;	
(ii)	$N(B)$	$3^{1+4} \cdot 2U_4(2) \cdot 2$;	
(iii)	$N(C)$	$(3 \times A_9) \cdot 2$;	
(iv)	$N(AADD)$	$3^2 \cdot U_4(3) \cdot D_8$;	
(v)	N(3 commuting A's)		soluble of order $2^7 \cdot 3^9$;
(vi)	N(12 commuting A's)		$3^6 \cdot 2M_{12}$;
(vii)	$N(BBBB_1)$		soluble of order $2^5 \cdot 3^9$;
(viii)	$N(BBBB_2)$		soluble of order $2^7 \cdot 3^8$.

Note. Here all shapes and orders are given in $\cdot 1$.

5-LOCAL STRUCTURE OF $\cdot O$

Element	Trace	Centralizer order in $\cdot O$	Shape in $\cdot 1$	Dimension of fixed space	Fixed space mod 2
A	-6	$2^8 \cdot 3^3 \cdot 5^3 \cdot 7$	$5 \times HJ$	0	—
B	-1	$2^4 \cdot 3 \cdot 5^4$	$5^{1+2} \cdot 2A_5$	4	$2^5 \cdot 3^{10}$
C	4	$2^6 \cdot 3^2 \cdot 5^3$	$5 \times (A_5 \times A_5) \cdot 2$	8	$2^{60} \cdot 3^{120} \cdot X^{75}$

LEMMA 5.1. *There is a unique conjugacy class of elementary abelian subgroups of order* 5^3.

Proof. Follows easily from the shape of $C(B)$.

COROLLARY. *This* 5^3*-subgroup contains elements of each of the types* A, B, *and* C *and its normalizer is transitive on the sets of elements of each type.*

Proof. The Syl_5 subgroups of $C(A)$ and $C(C)$ are precisely this 5^3-subgroup and from $C(B)$ it also contains B-elements.

COROLLARY. *Such a* 5^3*-subgroup contains* 40 A's, 24 B's *and* 60 C's; so $|N(5^3)| = 5^3 \cdot 40 \cdot 12 = 2^5 \cdot 3 \cdot 5^4$.

Proof. The numbers of elements of each type are inversely proportional to their centralizers in $N(5^3)$. Thus $A : B : C = 1/12 : 1/20 : 1/8 = 40 : 24 : 60 = 10 : 6 : 15$.

We denote the six B-groups of order 5 by ∞, $0 \cdots 4$ and consider the 15 5^2-subgroups such as $\infty 0$, on which the normalizer is certainly transitive. Letting x be the number of such groups to which an X-element belongs we find:

$$6 \times 15 = 5 \times 6 + 15c + 10a \Rightarrow a = 3, c = 2$$

and so such a group has shape $AABBCC$.

We are now in a position to write the group additively. The 10 A 5-subgroups are:

$$\begin{aligned}
2b_\infty + b_0 &= 2b_1 - b_2 = -b_3 + 2b_4 \\
2b_\infty - b_0 &= b_1 - 2b_3 = b_4 - 2b_2 \\
2b_\infty - {b_1}^1 &= b_0 - 2b_3 = b_2 - 2b_4 \\
2b_\infty + b_1 &= 2b_0 - b_4 = 2b_2 - b_3 \\
2b_\infty - b_2 &= b_1 - 2b_4 = -2b_0 + b_3 \\
2b_\infty + b_2 &= -b_0 + 2b_1 = 2b_3 - b_4 \\
2b_\infty - b_3 &= -2b_0 + b_2 = -2b_1 + b_4 \\
2b_\infty + b_3 &= -b_0 + 2b_4 = -b_1 + 2b_2 \\
2b_\infty - b_4 &= b_0 - 2b_2 = -2b_1 + b_3 \\
2b_\infty + b_4 &= -{b_1}^1 + 2b_0 = -b_2 + 2b_3 .
\end{aligned}$$

These are normalized by:

$$\begin{aligned}
&(\infty)(01234) \\
&(\infty)(0)(14)(23) \\
&(b_1 - b_2\ b_4 - b_3)(b_\infty)(b_0 - b_0) \\
&(b_\infty b_0)(b_1 - b_4)(b_2)(b_3) \\
&(b_i 2b_i - b_i - 2b_i)
\end{aligned}$$

giving the group $5^3 \cdot GL_2(5)$.

LEMMA 5.2. *The above 5^3-subgroup contains* 15 *subgroups of shape $AABBCC$,* 6 *of shape $BCCCCC$, and* 10 *of shape $AAACCC$.*

Proof. By observation.

Now clearly $N(BCCCCC) \leqslant N(B)$; moreover $N(AABBCC) \leqslant N(5^3)$ since the decomposition of the 24-space into the sum of two 8-spaces and two 4-spaces determined by a $AABBCC$ refines uniquely into the sum of six 4-spaces corresponding to B-elements. $|N(AAACCC)| = 2^5 \cdot 3^2 \cdot 5^3$ and has the shape $(5^2 \times A_5) \cdot 4 \cdot S_3$.

LEMMA 5.3. *If an elementary abelian subgroup of order 5^2 is not conjugate to a subgroup of a 5^3-subgroup it consists entirely of B-elements; there exists one class of such purely B-subgroups of order 5^2.*

Proof. (a) Plainly follows from proof of corollary to Lemma 5.1.

(b) The extraspecial 5^{1+2} of $C(B)$ may be generated by $\langle b, x, y; b^5 = x^5 = y^5 = b \circ x = b \circ y = 1, x \circ y = b\rangle$. The adjunction of z, where $z^5 = b \circ z = x \circ z = 1, y \circ z = xb$, gives a Syl_5 subgroup of $\cdot O$. $\langle b, yz\rangle$ is contained in no elementary abelian 5^3 and so has the form stated. Further a consideration of the action of $N(B)$ on the candidates for yz shows that there is just one class of such subgroups.

$|N(BBBBBB)| = 2^4 \cdot 3 \cdot 5^3$ abd has shape $5^2 \cdot 4 \cdot L_2(5)$.

THEOREM 5.1. *The normalizer of any 5-subgroup of $\cdot O$ is conjugate to a subgroup of one of the following*:

(i)	$N(A)$	$\leqslant (A_5 \times HJ) \cdot 2$	(*see the Appendix*)
(ii)	$N(B)$	$5^{1+2} \cdot GL_2(5)$	
(iii)	$N(C)$	$5 \cdot 4 \times (A_5 \times A_5) \cdot 2$	
(iv)	$N(5^3)$	$5^3 \cdot GL_2(5)$	
(v)	$N(AAACCC)$	$(5^2 \times A_5) \cdot 4 \cdot S_3$	
(vi)	$N(BBBBBB)$	$5^2 \cdot 4 \cdot L_2(5)$	

Note. All shapes and orders are given in $\cdot 1$.

7-LOCAL STRUCTURE OF $\cdot O$

There are two classes of elements of order seven in $\cdot O$.

Element	Trace	Centralizer order in $\cdot O$	Shape in $\cdot 1$	Dimension of fixed space	Fixed space mod 2Λ $\cdot$
A	-4	$2^4 \cdot 3^2 \cdot 5 \cdot 7^2$	$7 \times At$	0	—
B	3	$2^4 \cdot 3 \cdot 7^2$	$7 \times L_2(7)$	6	$2^{21} \cdot 3^{28} \cdot X^{14}$

The Syl_7 subgroup of $\cdot O$ has shape $AAAABBBB$ and $N(7^2)$ has shape $7^2 \cdot 6 \cdot A_4$.

THEOREM 7.1. *The normalizer of any 7-subgroup of* $\cdot O$ *is conjugate to a subgroup of one of the following*:

(i) $N(A) \quad \leqslant (L_2(7) \times A_7) \cdot 2$ (*see the Appendix*)

(ii) $N(B) \quad \leqslant (L_2(7) \times A_7) \cdot 2$

(iii) $N(7^2) \quad 7^2 \cdot 6 \cdot A_4$

OTHER LOCAL STRUCTURE IN $\cdot O$

Other elements of prime order in $\cdot O$ are:

Element	Trace	Centralizer order in $\cdot O$	Shape in $\cdot 1$	Dimension of fixed space	Fixed space mod 2Λ $\cdot$
$11A$	2	$2^2 \cdot 3 \cdot 11$	$11 \times S_3$	4	$2^6 \cdot 3^6 \cdot X^3$
$13A$	-2	$2^3 \cdot 3 \cdot 13$	$13 \times A_4$	0	—
$23A/B$	1	$2 \cdot 23$	23	2	$2 \cdot 3 \cdot X$

$$N(11) \leqslant N(3A) = (3Sz) \cdot 2$$
$$N(13) \leqslant N(2^2) = (A_4 \times G_2(4)) \cdot 2$$
$$N(23) \leqslant \cdot 2.$$

So these primes yield no new maximal subgroups.

APPENDIX

We shall make extensive use of the so-called Suzuki chain of subgroups of $\cdot O$ discovered by Thompson and described in [6]. Thus there exist subgroups $2A_n$ for $n = 2, 3, \ldots, 9$, nested in the natural manner, which are normalized by the following subgroups:

$$\cdot O,\ 6Sz \cdot 2,\ 2(G_2(4) \times A_4) \cdot 2,\ 2(HJ \times A_5) \cdot 2,\ 2(U_3(3) \times A_6) \cdot 2,$$
$$2(L_2(7) \times A_7) \cdot 2,\ 2(A_4 \times A_8) \cdot 2,\ 2(C_3 \times A_9) \cdot 2.$$

Elements of Prime Order

($2B/2D$) Take as $2B$ element that involution which negates coefficients in the second two octads of the *MOG* trio. It fixes just 120 2-vectors consisting of $\binom{8}{2} \cdot 2$ of shape 4^2 and 64 of shape 2^8 both with their nonzero coefficients in the first *MOG* octad. To demonstrate that these vectors do indeed give the E_8 lattice we exhibit a possible choice for the Dynkin diagram (see Fig. A.1).

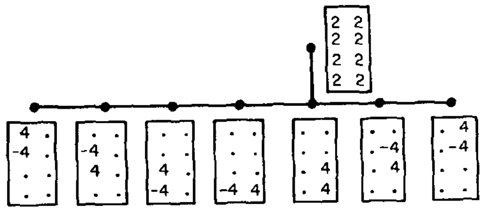


FIGURE A.1

(2C) The fact that a C-involution fixes a unique cross diametrically (p4) shows that $C._O(\epsilon_C) \leqslant N = E \cdot M_{24}$, where C is a dodecad, and so $C._O(\epsilon_C) \cong 2^{12} \cdot M_{12}$.

(4F) F-"involutions" are elements of $\cdot O$ of order four which square to -1. Elements of order four in the $2A_4$ mentioned above are of this class and $C._O(4F) = 2(2^2 \times G_2(4))$. Note that the centralizer in $\cdot 1$ of the image of a 4F is $(2^2 \times G_2(4)) \cdot 2$.

(3A) See [6, pp. 242, 244] for the subgroup $6Sz$ in the Suzuki chain, the ternary Golay code and the complex Leech Lattice.

(3B) See [1, p. 571] for identification and explicit lattice vectors; also Fig. 3.3 herein.

(3C) See [6, p. 242] for the subgroup $2(C_3 \times A_9) \cdot 2$ of the Suzuki chain.

(3D) See our work herein.

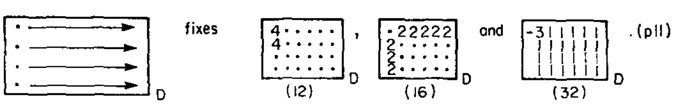
(5A) The 5-elements of the $2A_5$ in the Suzuki chain are of this class. The subgroup $2(A_5 \times HJ) \cdot 2$ contains the whole centralizer which is, therefore, of shape $2(5 \times HJ)$.

(5B) See [1, p. 567] for description of lattice.

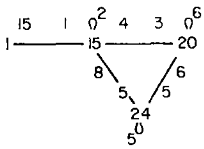
There are 30 ways in which an $\mathscr{S}$-lattice of type $2^5 \cdot 3^{10}$ can be extended to a lattice of type $2^{60} \cdot 3^{120} \cdot X^{75}$ fixed by a 5C. The corresponding 5C elements generate the subgroup

$$5^{1+2} \left(\frac{5^3 - 5}{4} = 30 \right).$$

(5C) An element of shape $1^4 \cdot 5^4$ in M_{24} belongs to this class; it fixes 60 2-vectors. Thus:



Joining vectors if they are orthogonal we obtain the following graph:



Dimensions :	1	16	18	25
Eigenvalues :	15	0	-5	3
	20	5	0	-4
	24	-6	4	0

We obtain this graph if we label the vertices with the 60 elements of the group A_5 and join x and y if, and only if, xy^{-1} is an involution. Incidence is preserved by right and left multiplication and outer automorphisms to give a group $(A_5 \times A_5) \cdot 2$. Note that the AM here does not interchange the two A_5s.

($7A/7B$) The Suzuki chain subgroup $2(L_2(7) \times A_7) \cdot 2$ has elements of class $7A$ in the $L_2(7)$ and $7B$ in the At ; the full centralizers of 7-elements of both classes are contained in it.

The assertion in the proof of Lemma 3.1 that in the subgroup $3 \times A_9$ no elements of the A_9 belong to the class $3C$ but all the diagonal 3-elements *are* in class $3C$ is due to Simon Norton. His argument is as follows: Restrict the 24-character to $3 \times 2A_9$ and observe that every faithful 8-character of $2A_9$ has the same nonzero value on all the elements of order 3, so the 24-character has nonzero value on the elements of order 3 in the $2A_9$ but is zero on the diagonal 3-elements.

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